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Set Logics and Their Representations

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We examine a set logic by means of all its representations as a concrete logic together with the automorphism groups of the representations. The most important are the minimal representations, i.e., the ones defined by minimal full collections of two-valued states. From this point of view we also investigate some Greechie diagrams.

1. DEFINITION AND BASIC PROPERTIES OF THE REPRESENTATIONS

Let E be an orthomodular poset (OMP) (Gudder, 1979; Kalmbach, 1983). Then E is referred to as a *set logic* provided $S_2(E)$ is full, $S_2(E)$ being the set of all two-valued finitely additive states on E. A subset S of $S_2(E)$ is said to be *full* if $x, y \in E$, $s(x) \le s(y) (s \in S) \Rightarrow x \le y$. A *concrete logic* (Sherstnev, 1968) is a couple $(\tilde{E}, X)$ where X is a set and $\tilde{E}$ is a collection of subsets of X satisfying:

1. $X \in \tilde{E}$.
2. $A \in \tilde{E} \Rightarrow X \backslash A \in \tilde{E}$.
3. $A, B \in \tilde{E}, A \cap B = \varnothing \Rightarrow A \cup B \in \tilde{E}$.

Proposition 1.1. (Gudder, 1979; Ptak and Pulmannova, 1991). An OMP E is isomorphic to a concrete logic iff E is a set logic.

We call every concrete logic isomorphic to E a *representation* for E. A representation $(\tilde{E}, X)$ is called *separating* (Navara and Tkadlec, 1991) if $\forall x, y \in X\ (x \ne y)\ \exists A \in \tilde{E}\ (x \in A$ and $y \notin A)$. It is clear that $(\tilde{E}, X)$ is separating

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iff the mapping $x \to \delta_x$ from X to $S_2(\tilde{E})$ defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

is injective. States of the form δ_x are referred to as *point states.* For example, if $S \subseteq S_2(E)$ is full, then we can obtain a separating representation for E if we put $X = S$ and $\tilde{E} = \{\tilde{e} | e \in E\}$, where $\tilde{e} = \{s \in S | s(e) = 1\}$. Conversely, if $(\tilde{E}, X)$ is a separating representation for E, then $S_p(\tilde{E})$ is full, $S_p(\tilde{E})$ being the set of all point states.

A representation $(\tilde{E}, X)$ is said to be *minimal* providing $S_p(\tilde{E})$ is a minimal (under inclusion) full collection of two-valued states. Obviously $(\tilde{E}, X)$ is minimal iff $\forall x \in X\ \exists A, B \in \tilde{E}\ (A \cap B = \{x\})$. A *trivial* representation for E is a representation $\tilde{E}$ satisfying $S_p(\tilde{E}) = S_2(E)$.

The representations $(\tilde{E}, X)$ and $(\tilde{F}, Y)$ are called *spatially isomorphic* if there exists a bijection $f: X \to Y$ such that f and f^{-1} are measurable, i.e., $\forall A \in \tilde{E}\ \forall B \in \tilde{F}\ [f(A) \in \tilde{F}$ and $f^{-1}(B) \in \tilde{E}]$.

Let $A(E)$ be the set of all atoms in E, E being an OMP. Then E is said to be *atomistic* if $\forall e \in E\ [e = \bigvee \{a \in A(E) | a \leq e\}]$.

Proposition 1.2. (Gudder, 1979; Ptak and Pulmannova, 1991). An atomistic OMP E is isomorphic to a concrete logic iff $\forall a, b \in A(E)$ $[a \not\perp b \Rightarrow \exists s \in S_2(E);\ s(a) = s(b) = 1]$.

Let Aut E denote the automorphism group of E. If E is a set logic and $(\tilde{E}, X)$ is its representation, then Aut E and Aut $\tilde{E}$ are isomorphic. An automorphism $h \in \text{Aut}\ \tilde{E}$ is said *to be carried by a point mapping* providing there exists $f: X \to X$ with $h(A) = f^{-1}(A)$ $(A \in \tilde{E})$. A representation $(\tilde{E}, X)$ is called *A-regular* if every $h \in \text{Aut}\ \tilde{E}$ is carried by a point mapping. It results from Navara and Tkadlec (1991) that the following statement is valid.

Proposition 1.3. A representation $\tilde{E}$ is A-regular iff $S_p(\tilde{E})$ is invariant under Aut $\tilde{E}$.

Definition 1.4. A *UR-logic* is a set logic which has only one representation (up to a spatial isomorphism). A *UMR-logic* is a set logic all of whose minimal representations are spatially isomorphic. We call a set logic E *A-regular* if its every minimal representation is A-regular. We call E *A-singular* in case the trivial representation for E alone is A-regular.

Let us give examples of UR-logics. Suppose $k, m \in \mathbf{N}$, $k \geq 2$ $m \geq 3$, and X is a set with card $X = km$. Then $X(km, k) = \{A \subset X | \text{card}\ A$ is a multiple of $k\}$ is a UR-logic (Sultanbekov, 1991).

A representation $(\tilde{E}, X)$ is called *regular* if every finitely additive signed measure on $\tilde{E}$ can be extended to a finitely additive signed measure on the algebra [say, $a(\tilde{E})$] of subsets of X generated by $\tilde{E}$. Let us denote by $V(\tilde{E})$

the vector space of all finitely additive signed measures on $\tilde{E}$. A *polar* (Ovchinnikov, 1991) of $\tilde{E}$ is defined by $\tilde{E}^0 = \{\mu \in V(a(\tilde{E})) | \forall \in \tilde{E}(\mu A = 0)\}$. The following theorem turns out to be very helpful.

Theorem 1.5 (Ovchinnikov, 1991). Let E be a finite set logic. A representation $(\tilde{E}, X)$ is regular iff $\dim \tilde{E}^0 + \dim V(\tilde{E}) \leq \operatorname{card} A(a(\tilde{E}))$.

Proof. Observe that $a(\tilde{E})$ can be obtained from $\tilde{E}$ by two extensions:

$$(\tilde{E})^\cap = \left\{ \bigcap_{i=1}^{k} A_i | k \in \mathbf{N}, A_i \in \tilde{E} \right\}$$

and

$$a(\tilde{E}) = (\tilde{E}^\cap)^\cup = \left\{ \bigcup_{j=1}^{k} B_j | k \in \mathbf{N}, B_j \in \tilde{E}^\cap \right\}$$

Since E is finite, it follows that $a(\tilde{E})$ is also finite. It is easy to prove that any finite algebra of subsets of X can be generated by a finite partition of X. Suppose $X = \bigcup \{X_i | i = 1, \ldots, n\}$, $X_i \cap X_j = \varnothing (i \neq j)$, $X_i \neq \varnothing (i = 1, \ldots, n)$, and $\{X_1, \ldots, X_n$ generates $a(\tilde{E})$. Then $A(a(\tilde{E})) = \{X_1, \ldots, X_n\}$. Since every $\mu \in V(a(\tilde{E}))$ is defined by its values on the atoms, we get $\dim V(a(\tilde{E})) = n$. Consider a linear mapping L: $V(a(\tilde{E})) \to V(\tilde{E})$ defined by $L(\mu) = \mu|_{\tilde{E}}$. Obviously $\operatorname{Ker} L = \tilde{E}^0$. Clearly $\tilde{E}$ is regular (i.e., $\operatorname{Im} L = V(\tilde{E})$] iff $\dim \operatorname{Im} L \geq \dim V(\tilde{E})$. The latter is valid iff

$$\dim V(\tilde{E}) + \dim \tilde{E}^0 \leq \dim \operatorname{Im} L + \dim \operatorname{Ker} L = \dim V(a(\tilde{E})) = n$$

Remark 1.6. As is obvious from the above proof, in Theorem 1.5, we may replace the inequality by the equality.

The regularity (A-regularity) of representations is invariant under spatial isomorphisms. At the same time, in general it is not invariant under arbitrary isomorphisms. That stimulates us to give the following definitions.

Definition 1.7. A set logic E is called *absolutely regular* if its every representation is regular. It is called *singular* provided its every representation is not regular.

Clearly E is absolutely regular iff its every minimal representation is regular and is singular if its trivial representation is not regular.

2. MINIMAL REGULAR REPRESENTATIONS FOR SOME GREECHIE DIAGRAMS

Denote by E_n ($n \geq 4$) the OMP whose Greechie diagram (Greechie, 1971; Gudder, 1979) is an n-polygon [in Kalmbach (1983) it is called a *loop*] which has three atoms on each edge. We suppose the polygon to be proper. Let us denote by $P_0, P_1, \ldots, P_{n-1}$ the vertices of the n-polygon. Denote by Q_i the middle atom between P_i and P_{i+1}. By L_n we denote the OMP whose Greechie diagram can be obtained from E_n by deleting Q_{n-1}. These atomistic OMPs satisfy the requirements of Proposition 1.2 and thus are set logics. We denote by Ext $S(E)$ the set of all extreme points of $S(E)$, $S(E)$ being the set of all states on E. Next, $l_n = \operatorname{card} S_2(L_n)$ and $e_n = \operatorname{card} S_2(E_n)$.

Remark 2.1. A state on E_n or L_n is obviously defined by its values on $P_0, P_1, \ldots, P_{n-1}$. For two-valued states we will list the vertices evaluated to 1 alone.

Theorem 2.2. (1) The generators of Aut E_n are t and q_0, where $t(P_i) = P_{i+1}$, $t(Q_i) = Q_{i+1}$ (a translation), $q_0(P_i) = P_{-i}$, and $q_0(Q_i) = Q_{-i}$ (a symmetry). The generators of Aut L_n are q and r, where $q(P_i) = P_{n-i-1}$, $q(Q_i) = Q_{n-i-1}$, r transposes P_0 and Q_0 and leaves invariant the other atoms (all indices are modulo n).

(2) Ext $S(L_n) = S_2(L_n)$, Ext $S(E_{2k}) = S_2(E_{2k})$, and Ext $S(E_{2k+1}) = S_2(E_{2k+1}) \cup \{e\}$, where $e(P_i) = 0.5$ $(i = 0, \ldots, 2k)$.

(3) l_n and e_n form Fibonacci sequences with $l_1 = 2$, $l_2 = 3$, $e_1 = 1$, and $e_2 = 3$.

Moreover,

$$e_n = l_{n-1} + l_{n-3} = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$l_n = \frac{2+\sqrt{5}}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \frac{\sqrt{5}-2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$

Proof. (1) Since any automorphism is obviously defined by its values on vertices and neighboring vertices are carried to neighboring ones, the assertion for Aut E_n follows. As to Aut L_n, it suffices to notice that $P_1, P_2, \ldots, P_{n-2}$ have to be evaluated to $P_1, P_2, \ldots, P_{n-2}$ or $P_{n-2}, P_{n-3}, \ldots, P_1$ and the action of Aut L_n on $\{P_0, Q_0, P_{n-1}, Q_{n-2}\}$ is transitive.

(2) It was proved in Ovchinnikov (1985) that Ext $S(E_{2k}) = S_2(E_{2k})$ and $e \in$ Ext $S(E_{2k+1})$. The rest is straightforward.

(3) There is a natural bijective correspondence between elements of $S_2(E_n)$ and subsets of $\{P_0, P_1, \ldots, P_{n-1}\}$ containing no neighbors. The assertion follows from Aigner (1979) and Vorobjov (1978).

Theorem 2.3. (1) If $n \geq 7$, then E_n admits a nonregular, minimal, and A-regular representation. The OMPs E_4, E_5 and E_6 are absolutely regular.

(2) The set logic L_6 is A-singular.

Proof. (1) Denote by X the set of all elements of $S_2(E_n)$ that equal 1 on two vertices alone. Let us show X to be suitable, i.e., $\tilde{E} = \{\tilde{P}_l | l = 0, \ldots, n-1\} \cup \{\tilde{Q}_l | l = 0, \ldots, n-1\}$, where $\tilde{P}_l = \{s \in X | s(P_l) = 1\}$ and $\tilde{Q}_l = \{s \in X | s(Q_l) = 1\}$ is the required representation. The set X can be separated into orbits, X_k, $k = 2, 3, \ldots, [n/2]$, under Aut E_n, and X_k consists of all elements of X whose central angle between the vertices evaluated to 1 equals $2\pi k/n$. Obviously card $X_k = n$ ($k = 2, 3, \ldots, [n/2]$) for an odd n and card $X_k = n$ ($k = 2, 3, \ldots, n/2 - 1$) and card $X_{n/2} = n/2$ for an even n. Since all $\tilde{P}_l \cap \tilde{P}_m$ ($|l - m| > 1$) are singletons, it follows that E is minimal. Let us verify that $\tilde{Q}_l \cap \tilde{Q}_m \neq \emptyset$. The required $s \in \tilde{Q}_l \cap \tilde{Q}_m$ can be constructed as follows. Suppose the diameter orthogonal to the chord between Q_l and Q_m meets E_n in the atom R opposite the chord. Then $s \in X_2$ whose central angle covers R is suitable. Finally, $\tilde{P}_l \cap \tilde{Q}_m \neq \emptyset$ ($|l - m| > 1$): take $a \in X_2$ with $s(P_l) = 1$ whose central angle does not cover Q_m. Then $s \in \tilde{P}_l \cap \tilde{Q}_m$.

Obviously X is invariant under Aut E_n. Hence by Proposition 1.3, $\tilde{E}$ is A-regular.

Every vertex $\tilde{\mathrm{P}}_l$ contains two states from each X_k provided n is odd. In case n is even $\tilde{P}_l$ contains two states from X_k ($k = 2, \ldots, n/2 - 1$) and one state from $X_{n/2}$. Suppose $\tilde{E}$ is regular. Then every signed measure μ on $\tilde{E}$ is defined by a suitable $f: X \to \mathbf{R}$ as follows:

$$\mu(A) = \sum_{x \in A} f(x) \qquad (A \in \tilde{E})$$

Consider the state μ on E_n defined by $\mu(P_l) = 0$ ($l = 0, \ldots, 1$). Then

$$0 = \sum_{l=0}^{n-1} \mu(P_l) = 2 \sum_{x \in X_2} f(x) + 2 \sum_{x \in X_3} f(x) + \cdots + 2 \sum_{x \in X[n/2]} f(x)$$

$$= 2 \sum_{x \in X} f(x) = 2\mu(X) = 2$$

This is a contradiction.

Let us now show E_6 to be absolutely regular. Define two-valued states a_0, d_0, b_0, c_0, and e by $a_0(P_1) = a_0(P_5) = 1$, $d_0(P_0) = d_0(P_3) = 1$, $b_0(P_0) = 1$, $c_0(P_0) = c_0(P_2) = c_0(P_4) = 1$, and $e(P_l) = 0$ ($l = 0, \ldots, 5$). Let a_l, d_l, b_l, and c_l ($l = 1, \ldots, 5$) be products of a_0, d_0, b_0, and c_0 with t^{-l}, t being the automorphism from Theorem 2.2. Then $\mathbf{a} = \{a_0, a_1, \ldots, a_5\}$, $\mathbf{b} = \{b_0, b_1, \ldots, b_5\}$, $d = \{d_0, d_1, d_2\}$, $\mathbf{c} = \{c_0, c_1\}$, and $\{e\}$ exhaust all the orbits in $S_2(E_6)$ under Aut E_6. For each couple (l, m), put $\tilde{P}(l, m) = \{s \in S_2(E_6 | s(P_l) = s(P_m) = 1\}$, $\tilde{Q}(l, m) = \{s \in S_2(E_6) | s(Q_l) = s(Q_m) = 1\}$ and $\widetilde{PQ}(l, m) = \{s \in S_2(E_6) | s(P_l) = s(Q_m) = 1\}$. Obviously $X \subset S_2(E_6)$ is full iff

$$X \cap \tilde{P}(l, m) \neq \emptyset, \qquad X \cap \tilde{Q}(l, m) \neq \emptyset, \qquad X \cap \widetilde{PQ}(l, m) \neq \emptyset$$

for all suitable l, m. Making use of this criterion let us describe up to a spatial isomorphism all minimal representations for E_6. Since $\tilde{P}(l, l+3) = \{d_l\}$, it follows that every representation contains $\mathbf{d}$. For brevity we write a_{024} instead of $\{a_0, a_2, a_4\}$. Then full sets of two-valued states for the minimal representations are the following:

(i) $\mathbf{a} \cup \mathbf{d} \cup \{e\}, \mathbf{d} \cup a_{024} \cup b_{024} \cup \{c_0\}$.

(ii) $\mathbf{b} \cup \mathbf{c} \cup \mathbf{d}$

(iii) $\mathbf{a} \cup \mathbf{d} \cup b_{024}, a_{01234} \cup \mathbf{d} \cup b_{04} \cup \{c_0, e\}, b_{01234} \cup \mathbf{d} \cup a_{04} \cup \mathbf{c}$

(iv) $\mathbf{a} \cup \mathbf{d} \cup b_{0134}, a_{0123} \cup \mathbf{d} \cup b_{0345} \cup \mathbf{c}, a_{0134} \cup \mathbf{d} \cup b_{0134} \cup \mathbf{c}, a_{01234} \cup \mathbf{d} \cup b_{0134} \cup \{c_0\}$.

Thus only two of ten minimal representations are A-regular. They are $\mathbf{a} \cup \mathbf{d} \cup \{e\}$ and $\mathbf{b} \cup \mathbf{d} \cup \mathbf{c}$. It is straightforward that all the aforementioned ten representations are regular.

(2) We have $S_2(L_6) = S_2(E_6) \cup \{f_0, f_1, f_2\}$, where f_0 equals 1 in vertices P_0, P_3, and P_5, f_1 in P_0 and P_5, and f_2 in P_0, P_2, and P_5. By the table of products of two-valued states with generators of Aut L_6, we obtain all orbits in $S_2(L_6)$ under Aut L_6:

$$a_{14} \cup b_{23} \cup d_{02} \cup f_{02}; \qquad b_{05} \cup \{e, f_1\}; \qquad a_{05} \cup b_{14}; \qquad a_{23} \cup c_{01}; \qquad \{d_1\}$$

Next, $\tilde{P}(1,3) = \{a_2, c_1\}$, $\tilde{P}(1,4) = \{d_1\}$, $\widetilde{PQ}(1,3) = \{a_0, b_1\}$, $\widetilde{PQ}(2,3) = \{a_1, d_2, f_2\}$, and $\tilde{Q}(1,3) = b_{05} \cup \{e, f_1\}$. Therefore, if $(\tilde{E}, X)$ is an arbitrary representation, then X has a nonempty intersection with each orbit listed above. Thus, if $\tilde{E}$ is in addition A-regular, then X contains all these orbits. Hence $X = S_2(L_6)$ and $\tilde{E}$ is trivial. The theorem follows.

Theorem 2.4. There exists a minimal, A-regular and regular representation $(\tilde{E}, X)$ for E_n with card $X = \lambda n$, $1 < \lambda \le 2 + 1/n$. Moreover, there exists a numeration of elements of the orbits in X under Aut $\tilde{E}$ such that the generators of Aut $\tilde{E}$ are carried by the point mappings $x_i \mapsto x_{i+1}$ and $x_i \mapsto x_{-i}$.

Proof. According to Proposition 1.3, X needs to be a union of orbits.

Case 1. E_{2k+1} $(k \ge 4)$; $\lambda = 2 + 1/(2k+1)$. Let a_0 be the two-valued state with $a_0(P_i) = 1$ for $i = 0, 2, 4, \ldots, 2k-2$, and b_0 be the one for $i = 0, 2, 4, \ldots, 2k-4$. Put $a_j = a_0 \cdot t^{-j}$ and $b_j = b_0 \cdot t^{-j}$ $(j = 1, \ldots, 2k)$. Since a_0 and b_0 are symmetric with respect to certain diameters, it follows that $\mathbf{a} = \{a_0, \ldots, a_{2k}\}$ and $\mathbf{b} = \{b_0, \ldots, b_{2k}\}$ are orbits. In what follows indices for a_j and b_j are taken modulo $2k+1$. Let c denote the state evaluating any vertex to 0. Put $X = \mathbf{a} \cup \mathbf{b} \cup \{c\}$. Then we get card $X = 4k+3$.

Define $T: X \to X$ by $Ta_j = a_{j+1}$, $Tb_j = b_{j+1}$, and $Tc = c$. Consider a concrete logic $\tilde{E}$ on X with $\tilde{P}_0 = a_{035\cdots 2k-1} \cup b_{057\cdots 2k-1}$, $\tilde{Q}_0 = b_{234} \cup \{a_2, c\}$,

$\tilde{P}_n = T^n(\tilde{P}_0)$, and $\tilde{Q}_n = T^n(\tilde{Q}_0)$ $(n = 1, \ldots, 2k)$ as atoms. To prove $\tilde{E}$ to be a representation for E_{2k+1} it suffices to verify that $\tilde{Q}_0 \cap \tilde{Q}_n \neq \varnothing$ $(n = 1, \ldots, 2k)$, $\tilde{P}_0 \cap \tilde{P}_n \neq \varnothing$ $(n = 2, \ldots, 2k-1)$, and $\tilde{P}_0 \cap \tilde{Q}_n \neq \varnothing$ $(n = 1, \ldots, 2k-1)$.

We have $\tilde{P}_n = a_{n,3+n,\ldots,2k-1+n} \cup b_{n,5+n,\ldots,2k-1+n}$. If $n \in \{2, \ldots, 2k-1\}$ is even, then $a_{3+n} \in \tilde{P}_0 \cap \tilde{P}_n$. If n is odd, then $a_n \in \tilde{P} \cap \tilde{P}_n$. We have $\tilde{Q}_n = b_{2+n,3+n,4+n} \cup \{a_{2+n}, c\}$. If $n \in \{1, 2, \ldots, 2k-1\}$ is odd, then $a_{2+n} \in \tilde{P}_0 \cap \tilde{Q}_n$. If n is even, then $b_{3+n} \in \tilde{P}_0 \cap \tilde{Q}_n$. Finally, $c \in \tilde{Q}_0 \cap \tilde{Q}_n$ for arbitrary n.

The minimality of the representation follows from $\tilde{P}_0 \cap \tilde{P}_3 = \{a_3\}$, $\tilde{P}_0 \cap \tilde{Q}_2 = \{b_5\}$, and $\tilde{Q}_0 \cap \tilde{Q}_3 = \{c\}$.

Let us prove the representation to be regular. Since the representation is minimal, it follows that $a(\tilde{E})$ consists of all the subsets of X. Therefore, by Theorem 1.5, it suffices to show that

$$\dim \tilde{E}^0 \leq \text{card } X - \dim V(\tilde{E}) = 4k + 3 - (2k+2) = 2k+1$$

Every $\mu \in V(a(\tilde{E}))$ is defined by $\alpha_j = \mu(a_j)$, $\beta_j = \mu(b_j)$, and $\gamma = \mu(c)$. By the definition of a polar, we have

$$\sum_{j=0}^{2k} \alpha_j + \sum_{j=0}^{2k} \beta_j + \gamma = 0 \tag{1}$$

and

$$\alpha_j + \alpha_{3+j} + \cdots + \alpha_{2k-1+j} + \beta_j + \beta_{j+5} + \cdots + \beta_{2k-1+j} = 0$$
$$(j = 0, \ldots, 2k) \tag{2}$$

and

$$a_j + \beta_j + \beta_{j+1} + \beta_{j+2} + \gamma = 0 \qquad (j = 0, \ldots, 2k) \tag{3}$$

Summing the equations (2), we obtain $k \sum_j \alpha_j + (k-1) \sum_j B_j = 0$. By (1), $\gamma = -(1/k) \sum \beta_j$. Therefore, (3) implies $\gamma, \alpha_j \in \text{lin}\{\beta_0, \ldots, \beta_{2k}\}$ $(j = 0, \ldots, 2k)$. Thus we get $\dim \tilde{E}^0 \leq \dim \text{lin}\{\beta_0, \ldots, \beta_{2k}\} \leq 2k+1$.

Case 2. E_{2k} $(k \geq 5)$; $\lambda = 3/2$. Put $A_m = \{k \geq 5 | k \not\equiv 0 \pmod{4m-1}$ and $k \equiv 0 \pmod{4r-1}$, $\forall r \in \mathbf{N}, r \leq m-1\}$. Then $\{A_m | m \in \mathbf{N}\}$ is a partition of $\{k | k \in \mathbf{N}$ and $k \geq 5\}$. Observe that if $k \neq 6$, then

$$\forall m \in \mathbf{N} \; (k \in A_m \Rightarrow k \geq 4m+1) \tag{*}$$

Let a_0 be a two-valued state defined by $a_0(P_i) = 1$ $(i = 0, 2, 4, \ldots, 2m-2, 2m+1, 2m+3, 2m+5, \ldots, 2k-(2m+1), 2k-(2m-2), 2k-2m, \ldots, 2k-4, 2k-2)$. Let d_0 be a two-valued state with $d_0(P_0) = d_0(P_k) = 1$. As in Case 1, making use of t, we get two orbits of Aut E_{2k}: $\mathbf{a} = a_{012\cdots 2k-1}$ and $\mathbf{d} = d_{012\cdots k-1}$; for a_i (d_i) the indices are taken modulo $2k$ (k). Put $X = \mathbf{a} \cup \mathbf{d}$.

Then card $X = 3k$. Let T: $X \to X$ be defined by $Ta_i = a_{i+1}$ and $Td_i = d_{i+1}$. Put

$$\tilde{P}_0 = a_{02\cdots 2m-2,\, 2m+1,\ldots,2k-2m-1,2k-2m+2,\ldots,2k-2} \cup \{d_0\}$$

and

$$\tilde{Q}_0 = a_{2m,2k-2m+1} \cup d_{23\cdots k-1}$$

$\tilde{P}_l$ and $\tilde{Q}_l$ are defined as above.

Let us demonstrate that $\tilde{Q}_0 \cap \tilde{Q}_l \cap \mathbf{d} \neq \varnothing$. Since card $\tilde{Q}_i \cap \mathbf{d} = k-2$ $(i \geq 0)$, it follows that $\{2, 3, \ldots, k-1\} \cap \{2+l, 3+l, \ldots, k-1+l\} = \varnothing$ implies card$\{2, 3, \ldots, k-1\} \cup \{2+l, 3+l, \ldots, k-1+l\} = 2k-4 \geq k+1$. Thus there exist $n \in \{2, 3, \ldots, k-1\}$ and $p \in \{2+l, 3+l, \ldots, k-1+l\}$ satisfying $n \equiv p \pmod k$.

Let us now show that $\tilde{P}_0 \cap \tilde{Q}_l \neq \varnothing$ $(l = 1, \ldots, 2k-2)$. We have

$$\tilde{Q}_l = a_{2m+l,2k-2m+1+l} \cup d_{2+l,3+l,\ldots,k-1+l}.$$

Obviously if $l \neq k-1$ and $l \neq k$, then there exists $p \in \{2+l, 3+l, \ldots, k-1+l\}$ such that $p \equiv 0 \pmod k$ and therefore $d_0 \in \tilde{P}_0 \cap \tilde{Q}_l$. If $l = k-1$ or $l = k$, then by (*), we get $(2m+k-1, k-2m\} \cap \{2m+1, 2m+3, \ldots, 2k-2m-1\} \neq \varnothing$ and $\{2m+k, k-2m+1\} \cap \{2m+1, 2m+3, \ldots, 2k-2m-1\} \neq \varnothing$. Therefore, $\tilde{P}_0 \cap \tilde{Q}_l \cap \mathbf{a} \neq \varnothing$. In the case $k = 6$ and $m = 2$ the proof is straightforward.

Consider

$$\tilde{P}_l = a_{l,2+l,\ldots,2m-2+l,2m+1+l,\ldots,2k-2m-1+l,2k-2m+2+l,\ldots,2k-2+l} \cup \{d_l\}$$

$$(2 \leq l \leq 2k-2)$$

If $l \in \{3, 5, \ldots, 2m+1\}$, then $\tilde{P}_0 \cap \tilde{P}_l \cap a_{l,2+l,\ldots,2m-2+l} \neq \varnothing$. If $l \in \{2m+3, 2m+5, \ldots, 2k-3\}$, then $\tilde{P}_0 \cap \tilde{P}_l \cap a_{2k-2m+2+l,\ldots,2k-2+l} \neq \varnothing$. Suppose l is even. If $l \in \{2, 4, \ldots, 2m-2\}$, then $a_l \in \tilde{P}_0 \cap \tilde{P}_l$. If $l \in \{2m, 2m+2, \ldots, 4m-4\}$, then $a_{2k-2m+2+l} \in \tilde{P}_0 \cap \tilde{P}_l$. If $l = 4m-2$ or $l = 4m$, then we have $a_{2m+1+l} \in \tilde{P}_0 \cap \tilde{P}_l$ [use (*)]. Finally, if $l \in \{4m+2, 4m+4, \ldots, 2k-2\}$, then $a_{2k-2m-1+l} \in \tilde{P}_0 \cap \tilde{P}_l$. If $k = 6$ and $m = 2$, then the inequality $\tilde{P}_0 \cap \tilde{P}_l \neq \varnothing$ is straightforward.

The representation is minimal since $\tilde{P}_0 \cap \tilde{Q}_2 = \{d_0\}$ and $\tilde{P}_0 \cap \tilde{Q}_k = \{a_{k-2m+1}\}$ or $\tilde{P}_0 \cap \tilde{Q}_k = \{a_{2m+k}\}$.

Let us show the representation to be regular. Put $\alpha_i = \mu(a_i)$ and $\delta_i = \mu(d_i)$. For α_i (δ_i) the indices are considered modulo $2k$ (k). By the definition of a polar we obtain

$$\sum_{j=0}^{2k-1} \alpha_j + \sum_{j=0}^{k-1} \delta_j = 0 \tag{4}$$

$$\alpha_j + \alpha_{2+j} + \cdots + \alpha_{2m-2+j} + \alpha_{2m+1+j} + \cdots + \alpha_{2k-2m-1+j} + \cdots + \alpha_{2k-2+j} + \delta_j = 0 \quad (j = 0, \ldots, 2k-1) \tag{5}$$

$$\alpha_{2m+j} + \alpha_{2k-2m+1+j} + \delta_{2+j} + \cdots + \delta_{k-1+j} = 0 \qquad (j = 0, \ldots, 2k-1) \tag{6}$$

If we sum equations (5), then we obtain $(k-1)\sum_j \alpha_j + 2\sum_j \delta_j = 0$. By (4), this gives $\sum_j \alpha_j = \sum_j \delta_j = 0$. Therefore, the system (6) gets the form

$$\begin{cases} \alpha_0 + \alpha_{Am-1} = \delta_{2m-1} + \delta_{2m} \\ \alpha_1 + \alpha_{Am} = \delta_{2m} + \delta_{2m+1} \\ \ldots \\ \alpha_{2k-1} + \alpha_{Am-2} = \delta_{2m+2} + \delta_{2m-1} \end{cases} \tag{6'}$$

Since $k \in A_m$ and thus $k \not\equiv 0 \ (\mathrm{mod}(4m-1))$, it follows that if we take equations of (6′) with numbers $0, 4-1, 8m-2, \ldots$ (we consider the addition modulo $2k$), then we obtain all the equations of (6′). Therefore, we get $\alpha_j = (-1)^j \alpha_0 + g_j$ $(j = 1, 2, \ldots, 2k-1)$ with $g_j \in \mathrm{lin}[\delta_0, \delta_1, \ldots, \delta_{k-1}]$; then the first equation in (5) implies $m\alpha_0 - [k-1-(2m-1)]\alpha_0 + (m-1)\alpha_0 + \delta_0 + \sum g_l = 0$ and hence $(k+1-4m)\alpha_0 = \delta_0 + \sum g_l$. Thus for all $j = 0, \ldots, 2k-1$ we have $\alpha_j \in \mathrm{lin}\{\delta_0, \ldots, \delta_{k-1}\}$. Since $\sum_{j=0}^{k-1} \delta_j = 0$, we get

$$\dim \tilde{E}^0 \leq \dim \mathrm{lin}\{\delta_0, \ldots, \delta_{k-1}\} \leq k-1 = 3k-(2k+1) = \mathrm{card}\, X - \dim V(\tilde{E}).$$

Case 3. E_n $(n = 4, 5, 6, 7, 8)$. For E_4 or E_5 the required representation is obviously unique. We have already examined E_6 in Theorem 2.3. For E_7 the required representation is given by the two-valued states a_0 and b_0, where $a_0(P_i) = 1$ $(i = 0, 2, 5)$ and $b_0(P_0) = 1$. Finally, for E_8 it suffices to take $a_0(P_j) = 1$ $(j = 0, 3, 5)$, $d_0(P_0) = d_0(P_4) = 1$, and $c(P_j) = 0$ $(j = 0, \ldots, 7)$. The values for λ and card X are given in Table I.

The theorem follows.

Remark 2.5. For E_{6k+3} $(k \geq 1)$ we can also take $\lambda = 1 + 4/(6k+3)$. Consider the two-valued states defined by $a_0(P_i) = 1$ $(i = 0, 2, 4, \ldots, 6k-2)$, $b_0(P_i) = 1$ $(i \equiv 0 (\mathrm{mod}\ 3))$, and $c(P_i) = 0$ $(i = 0, \ldots, 6k+2)$.

Table I.

n	card X	λ
4	7	1.75
5	10	2
6	10	$1\frac{2}{3}$
7	14	2
8	13	1.625

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